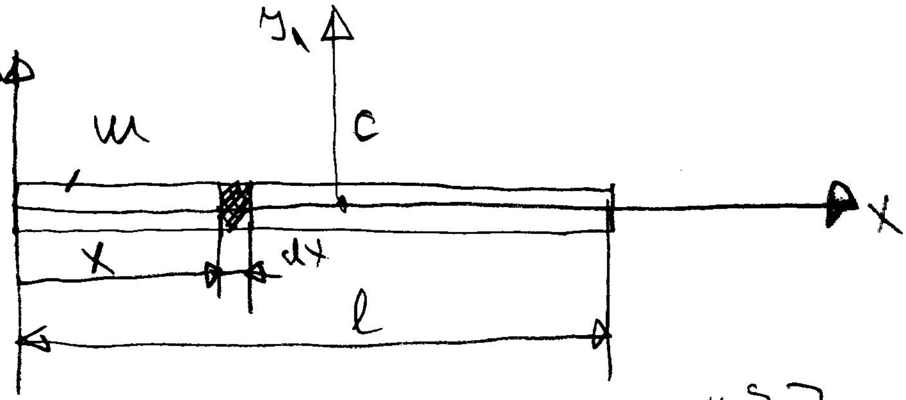
Геометрија маса

1) Штап



$x_C = \frac{1}{m}\int_{(m)} x\,dm$, $m = \rho l$, $\rho\left[\frac{kg}{m}\right]$

$dm = \rho\,dx$

$\boxed{x_C} = \frac{1}{\cancel{\rho} l}\int_0^l x\,\cancel{\rho}\,dx = \frac{1}{l}\frac{1}{2}x^2\Big|_0^l = \frac{1}{\cancel{l}}\frac{1}{2}l^{\cancel{2}} = \boxed{\frac{l}{2}}$ $C\left(\frac{l}{2}, 0\right)$

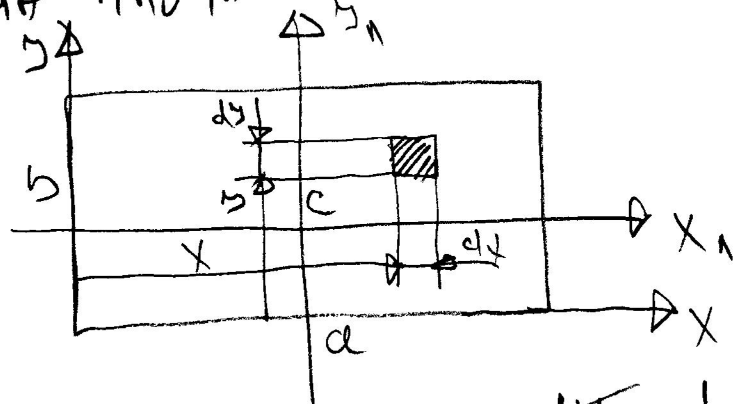
$\boxed{J_y} = \int_{(m)} x^2\,dm = \int_0^l x^2\rho\,dx = \rho\,\frac{1}{3}x^3\Big|_0^l = \frac{m}{3l}l^3 = \boxed{\frac{ml^2}{3}}$

$\boxed{J_{y1}} = J_y - m x_C^2 = \frac{ml^2}{3} - m\frac{l^2}{4} = \boxed{\frac{ml^2}{12}}$

2) Правоугаона плоча

$m = \rho ab$

$dm = \rho\,dx\,dy$



$x_C = \frac{1}{m}\int_{(m)} x\,dm = \frac{1}{m}\iint x\,\rho\,dx\,dy = \frac{\cancel{m}}{ab}\frac{1}{\cancel{m}}\int_0^a x\,dx\int_0^b dy =$

$= \frac{1}{ab}\frac{1}{2}x^2\Big|_0^a\,y\Big|_0^b = \frac{1}{\cancel{ab}}\frac{a^{\cancel{2}}\cancel{b}}{2} = \frac{a}{2}$

$y_C = \frac{1}{m}\int y\,dm = \dots = \frac{b}{2}$ $C\left(\frac{a}{2}, \frac{b}{2}\right)$

$$\boxed{J_y} = \int_{(m)} x^2\,dm = \iint x^2 \rho\,dx\,dy = \frac{m}{ab}\int_0^a x^2\,dx \int_0^b dy =$$

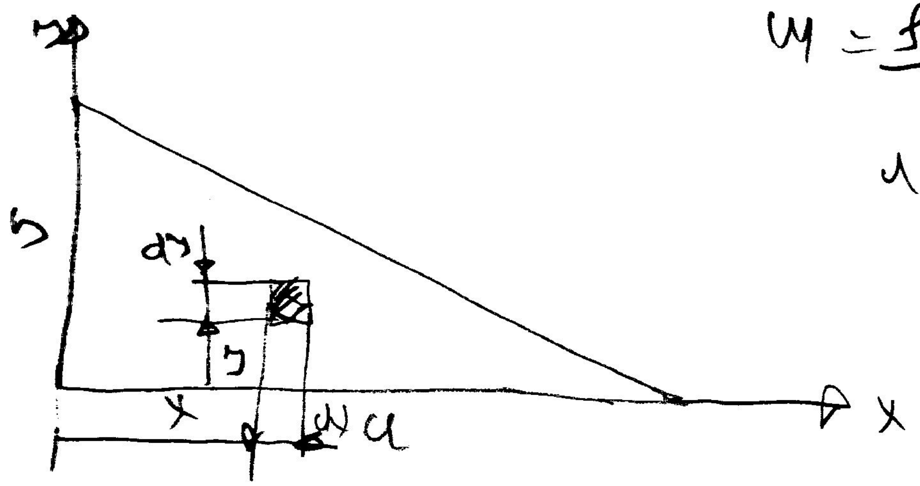
$$= \frac{m}{ab}\,\frac{1}{3}x^3\Big|_0^a\, y\Big|_0^b = \frac{m}{3\cancel{a}\cancel{b}}\,a^{\cancel{3}2}\,\cancel{b} = \boxed{\frac{ma^2}{3}}$$

$$J_{y_1} = J_y - m\left(\frac{a}{2}\right)^2 = \frac{ma^2}{3} - \frac{ma^2}{4} = \frac{ma^2}{12}$$

$$J_x = \int y^2\,dm = \iint y^2\rho\,dx\,dy = \ldots = \frac{mb^2}{3}$$

$$J_{x_1} = \ldots = \frac{mb^2}{12}, \quad J_z = \int (x^2+y^2)\,dm = \frac{m}{12}(a^2+b^2)$$

3) ТРОУГАЭНА ПЛОЧА



$$m = \frac{\rho ab}{2}$$

$$dm = \rho\,dx\,dy$$

$$x_c = \frac{1}{m}\int_{(m)} x\,dm = \frac{1}{m}\iint x\rho\,dx\,dy = \frac{2\cancel{m}}{ab}\frac{1}{\cancel{m}}\int_0^a x\,dx\int_0^{b(1-\frac{x}{a})} dy$$

$$\frac{x}{a} + \frac{y}{b} = 1 \Rightarrow y = b - \frac{b}{a}x = b\left(1 - \frac{1}{a}x\right)$$

$$y(x) = b\left(1 - \frac{x}{a}\right)$$

$$\boxed{x_c} = \frac{2}{ab}\int_0^a x\,dx\, y\Big|_0^{b(1-\frac{x}{a})} = \frac{2}{a\cancel{b}}\int_0^a \cancel{b}\left(1-\frac{x}{a}\right)x\,dx =$$

$$= \frac{2}{a}\left[\int_0^a x\,dx - \frac{1}{a}\int_0^a x^2\,dx\right] = \frac{2}{a}\left[\frac{x^2}{2}\Big|_0^a - \frac{1}{a}\frac{1}{3}x^3\Big|_0^a\right]$$

$$= \frac{2}{a}\left[\frac{a^2}{2} - \frac{a^2}{3}\right] = \frac{\cancel{2}}{\cancel{a}}\,\frac{a^{\cancel{2}}}{\cancel{6}_3} = \boxed{\frac{a}{3}}$$

$y_C = \frac{1}{m}\int y\,dm \ldots = \frac{b}{3} \qquad C\left(\frac{a}{3}, \frac{b}{3}\right)$

$$\boxed{J_y} = \int_{(m)} x^2\,dm = \iint x^2 \rho\,dx\,dy = \frac{2m}{ab}\int_0^a x^2\,dx \int_0^{b\left(1-\frac{x}{a}\right)} dy =$$

$$= \frac{2m}{ab}\int_0^a x^2\,dx\; y\Big|_0^{b\left(1-\frac{x}{a}\right)} = \frac{2m}{a\cancel{b}}\int x^2\,dx\,\cancel{b}\left(1-\frac{x}{a}\right) =$$

$$= \frac{2m}{a}\int_0^a x^2\left(1-\frac{x}{a}\right)dx = \frac{2m}{a}\left(\int_0^a x^2\,dx - \frac{1}{a}\int_0^a x^3\,dx\right) =$$

$$= \frac{2m}{a}\left(\frac{1}{3}x^3\Big|_0^a - \frac{1}{4a}x^4\Big|_0^a\right) =$$

$$= \frac{2m}{a}\left(\frac{a^3}{3} - \frac{a^3}{4}\right) = \frac{2m}{\cancel{a}}\,\frac{a^{\cancel{3}\,2}}{\cancel{12}_6} = \boxed{\frac{ma^2}{6}}$$

$$J_{y1} = J_y - m\left(\frac{a}{3}\right)^2 = \frac{ma^2}{6} - m\frac{a^2}{9} =$$

$$= \frac{ma^2}{18}$$

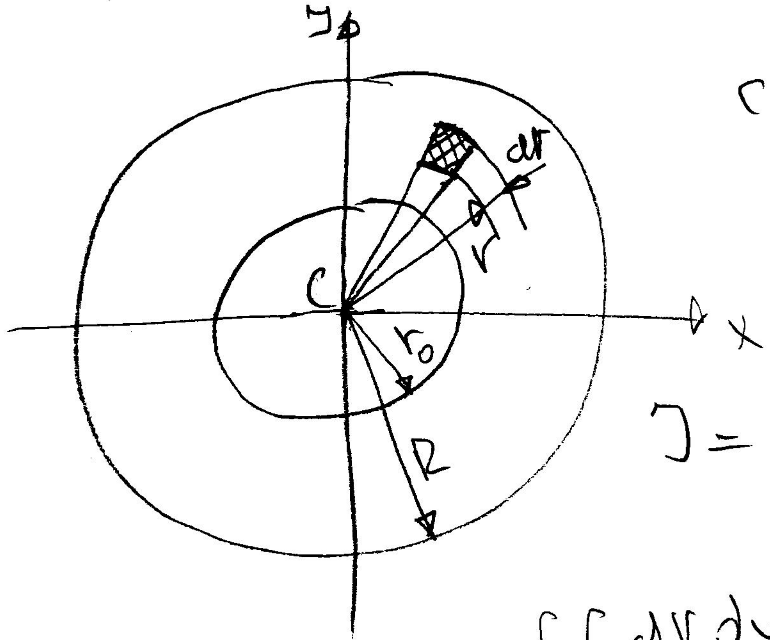
$$J_x = \int y^2\,dm = \ldots = \frac{mb^2}{6}$$

$$J_{x1} = \ldots = \frac{mb^2}{18}$$

$$J_z = \int (x^2+y^2)\,dm = \frac{m}{6}(a^2+b^2)$$

$$J_{z1} = \frac{m}{18}(a^2+b^2)$$

4) Плоча облика кружног прстена



$C(0,0)$

$x = r\cos\varphi$

$y = r\sin\varphi$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \varphi} & \frac{\partial y}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \cos\varphi & \sin\varphi \\ -r\sin\varphi & r\cos\varphi \end{vmatrix}$$

$$A = \int_{(A)} dA = \iint dx\,dy = \iint dr\,d\varphi\,|J| =$$

$$= \iint dr\,d\varphi\,r = \int_{r_0}^{R} r\,dr \int_0^{2\pi} d\varphi = \cancel{2}\pi \frac{1}{\cancel{2}} r^2 \Big|_{r_0}^{R} =$$

$$= (R^2 - r_0^2)\pi$$

$$m = \rho A = \rho(R^2 - r_0^2)\pi$$

$$dm = \rho\,dA = \rho\,r\,dr\,d\varphi$$

$$J_y = \int_{(m)} x^2\,dm = \iint r^2\cos^2\varphi\,\rho\,r\,dr\,d\varphi =$$

$$= \frac{m}{(R^2 - r_0^2)\pi} \int_{r_0}^{R} r^3\,dr \int_0^{2\pi} \cos^2\varphi\,d\varphi = (*)$$

$$\int_0^{2\pi} \cos^2\varphi\,d\varphi = \int_0^{2\pi} \frac{1+\cos 2\varphi}{2}\,d\varphi = \frac{1}{2}\left(\varphi\Big|_0^{2\pi} + \frac{1}{2}\sin 2\varphi\Big|_0^{2\pi}\right)$$

$$= \pi$$

$$(*)\ J_y = \frac{m\cancel{\pi}}{(R^2 - r_0^2)\cancel{\pi}} \frac{1}{4} r^4 \Big|_{r_0}^{R} = \frac{m}{4(R^2 - r_0^2)}(R^4 - r_0^4) =$$

$$= \frac{m}{4}(R^2 + r_0^2), \quad J_x = J_y = \frac{m}{4}(R^2 + r_0^2)$$

$$J_z = \int (x^2+y^2)\, dm = \frac{m}{2}(R^2 + r_0^2)$$

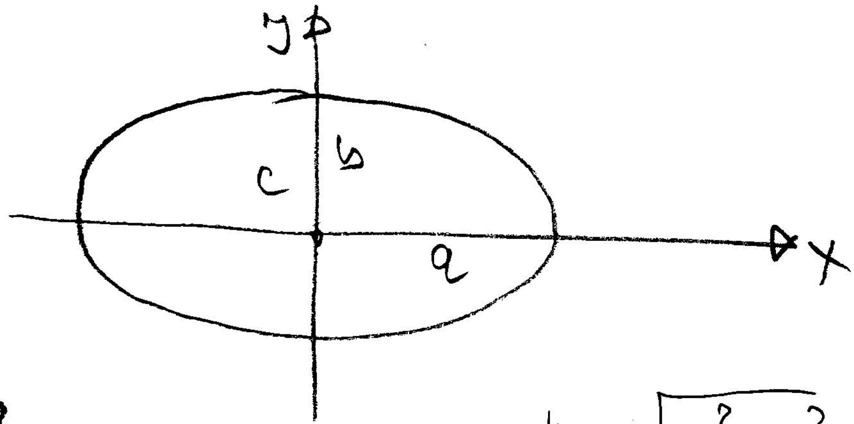
за $r_0 = 0$

$$J_x = J_y = \frac{m}{4}R^2, \quad J_z = \frac{m}{2}R^2.$$

за $r_0 \approx R$

$$J_x = J_y = \frac{m}{2}R^2, \quad J_z = mR^2$$

5) ПОВРШИНА ОБЛИКА ЕЛИПСЕ



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow y = \pm\frac{b}{a}\sqrt{a^2-x^2}$$

$$A = \iint dx\,dy = \int_{-a}^{a} dx \int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} dy = \int_{-a}^{a} dx\; y \Bigg|_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} =$$

$$= \int_{-a}^{a} dx\, 2\frac{b}{a}\sqrt{a^2-x^2} = 2\frac{b}{a}\int_{-a}^{a}\sqrt{a^2-x^2}\,dx =$$

$$= \left| \begin{array}{l} x = a\sin\varphi \\ dx = a\cos\varphi\, d\varphi \\ -\frac{\pi}{2} \le \varphi \le \frac{\pi}{2} \end{array} \right| = 2\frac{b}{\cancel{a}}\int_{-\pi/2}^{\pi/2} \cancel{a}\cos\varphi\, a\cos\varphi\, d\varphi$$

$$= 2ab\int_{-\pi/2}^{\pi/2}\cos^2\varphi\, d\varphi$$

$$A = \cancel{2}ab\int_{-\pi/2}^{\pi/2}\frac{1+\cos 2\varphi}{\cancel{2}}\,d\varphi = ab\left(\varphi\Big|_{-\pi/2}^{\pi/2} + \frac{1}{2}\sin 2\varphi\Big|_{-\pi/2}^{\pi/2}\right)$$

$$= ab\pi \quad , m = \rho A = \rho ab\pi$$

$$J_y = \int_{(m)} x^2\,dm = \iint x^2 \rho\,dx\,dy =$$

$$= \frac{m}{ab\pi}\int_{-a}^{a} x^2\,dx\int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} dy = \frac{m}{ab\pi}\int_{-a}^{a} x^2\,dx\,2\frac{b}{a}\sqrt{a^2-x^2}$$

$$= \frac{m}{a\cancel{b}\pi}\,2\frac{\cancel{b}}{a}\int_{-a}^{a} x^2\sqrt{a^2-x^2}\,dx =$$

$$= \frac{2m}{a^2\pi}\int_{-a}^{a} x^2\sqrt{a^2-x^2}\,dx = \left|\begin{array}{l} x = a\sin\varphi \\ dx = a\cos\varphi\,d\varphi \\ -\frac{\pi}{2} \le \varphi \le \frac{\pi}{2}\end{array}\right|$$

$$= \frac{2m}{a^2\pi}\int_{-\pi/2}^{\pi/2} a^2\sin^2\varphi\; a\cos\varphi\; a\cos\varphi\,d\varphi =$$

$$= \frac{2m}{\cancel{a^2}\pi}\,a^{\cancel{4}\,2}\int_{-\pi/2}^{\pi/2}\sin^2\varphi\cos^2\varphi\,d\varphi =$$

$$= \frac{\cancel{2}m\,a^2}{\pi}\int_{-\pi/2}^{\pi/2}\frac{1}{\cancel{4}\,2}(2\sin\varphi\cdot\cos\varphi)^2\,d\varphi =$$

$$= \frac{m a^2}{2\pi}\int_{-\pi/2}^{\pi/2}\sin^2 2\varphi\,d\varphi = \frac{m a^2}{2\pi}\int_{-\pi/2}^{\pi/2}\frac{1-\cos 4\varphi}{2}\,d\varphi$$

$$= \frac{m a^2}{4\pi}\left(\varphi\Big|_{-\pi/2}^{\pi/2} - \frac{1}{4}\sin 4\varphi\Big|_{-\pi/2}^{\pi/2}\right)$$

$$J_y = \frac{m a^2}{4\pi}\pi = \frac{m a^2}{4}$$

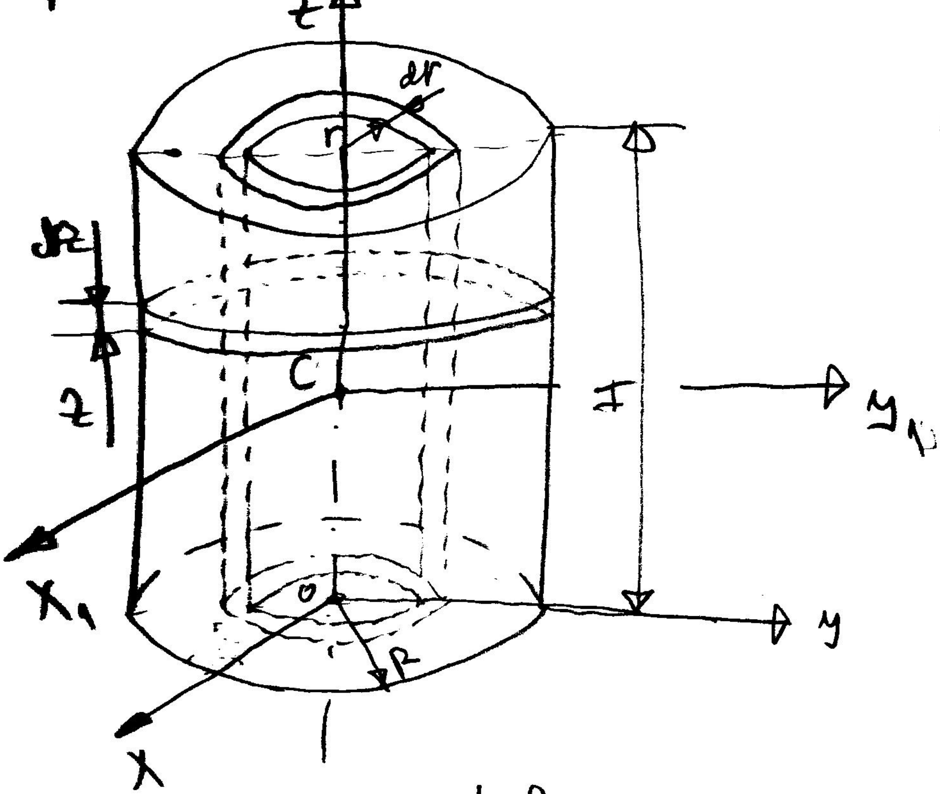
$$J_x = \int y^2 \, dm = \ldots = \frac{m b^2}{4}$$

$$J_z = \frac{m}{4}\left(a^2 + b^2\right)$$

б) Цилиндрична цев

$m = \rho V$, $\rho \left[\frac{kg}{m^3}\right]$

$dm = \rho \, dV$



$$dV = (R^2 - r_0^2)\pi \, dz \quad \Big|\cdot\int$$

$$\boxed{V} = (R^2 - r_0^2)\pi z \Big|_0^H = \boxed{(R^2 - r_0^2)\pi H}$$

$$\boxed{z_C} = \frac{1}{m}\int_{(m)} z \, dm = \frac{1}{m}\int_{(V)} z \rho \, dV = \frac{1}{m}\frac{m}{V}\int_{(V)} z \, dV =$$

$$= \frac{1}{V}\int_0^H z (R^2 - r_0^2)\pi \, dz = \frac{(R^2 - r_0^2)\pi}{V}\int_0^H z \, dz =$$

$$= \frac{(R^2 - r_0^2)\pi}{V}\frac{1}{2}H^2 = \frac{\cancel{(R^2 - r_0^2)\pi}}{\cancel{(R^2 - r_0^2)\pi} H}\frac{1}{2}H^2 = \boxed{\frac{1}{2}H}$$

$$\boxed{J_z} = \int_{(m)} r^2 \, dm = \int_{(V)} r^2 \rho \, dV = \frac{m}{V} \int_{(V)} r^2 \, dV = \circledast$$

$$dV = [(r+dr)^2 - r^2]\pi H =$$

$$= (\cancel{r^2} + 2r\,dr + dr^2 \cancel{- r^2})\pi H = 2r\,dr\,\pi H$$

$$\circledast$$

$$= \frac{m}{V} \int 2r^3 \, dr\,\pi H = \frac{2m\pi H}{V} \int_{r_0}^{R} r^3 \, dr =$$

$$= \frac{\cancel{2}m \cancel{\pi H}}{(R^2 - r_0^2)\cancel{\pi H}} \frac{1}{\cancel{4}_2} r^4 \Big|_{r_0}^{R} = \frac{m}{(R^2 - r_0^2)} \frac{1}{2} (R^4 - r_0^4) =$$

$$= \frac{m}{2} (R^2 + r_0^2)$$

$$J_{xy} = \int_{(m)} z^2 \, dm = \int_{(V)} z^2 \rho \, dV = \frac{m}{V} \int_{(V)} z^2 \, dV =$$

$$= \frac{m}{V} \int_0^H z^2 (R^2 - r_0^2)\pi \, dz = \frac{m}{V} (R^2 - r_0^2)\pi \int_0^H z^2 \, dz$$

$$= \frac{m}{\cancel{(R^2 - r_0^2)\pi H}} \cancel{(R^2 - r_0^2)\pi} \frac{1}{3} z^3 \Big|_0^H = \frac{m}{3} H^2$$

$$J_{xz} = J_{yz}$$

$$J_z = J_{xz} + J_{yz} = 2J_{xz} \Rightarrow J_{xz} = \frac{1}{2} J_z = \frac{m}{4} (R^2 + r_0^2)$$

$$J_x = J_{xz} + J_{xy} = \frac{m}{4} (R^2 + r_0^2) + \frac{m}{3} H^2$$

$$J_{x1} = J_x - m\left(\frac{H}{2}\right)^2 = \frac{m}{4}(R^2 + r_0^2) + \frac{m}{3} H^2 - \frac{m}{4} H^2$$

$$J_{x1} = \frac{m}{4}(R^2+r_0^2) + \frac{m}{12}H^2 = \frac{m}{12}(3R^2+3r_0^2+H^2)$$

$$J_{y1} = J_{x1}$$

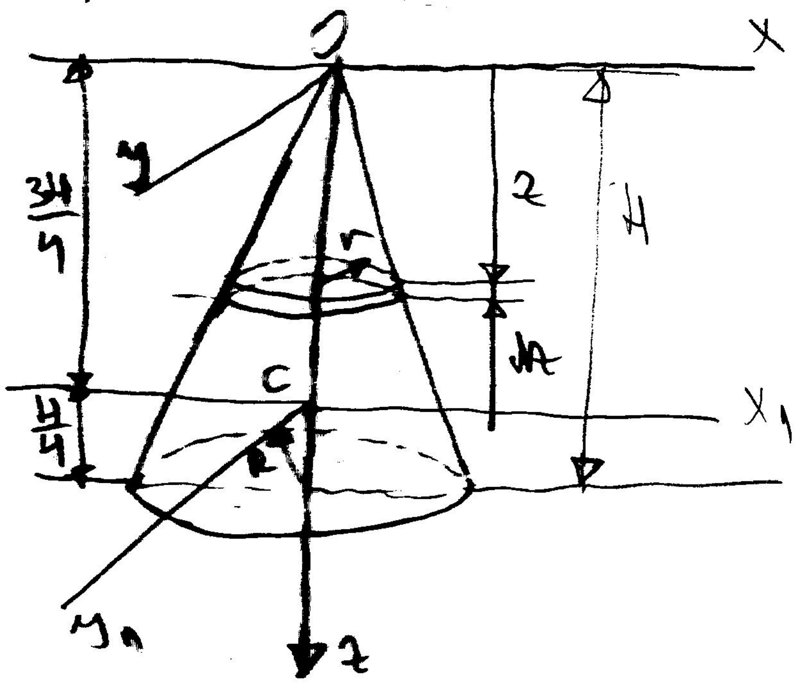
за $r_0 = 0$

$$J_z = \frac{m}{2}R^2, \quad J_{x1} = J_{y1} = \frac{m}{12}(3R^2+H^2)$$

за $r_0 = R$

$$J_z = mR^2, \quad J_{x1} = J_{y1} = \frac{m}{12}(6R^2+H^2)$$

7) Прав кружни конус



$m = \rho V$, $\rho \left[\frac{kg}{m^3}\right]$

$dm = \rho\, dV$

$$\frac{R}{H} = \frac{r}{z}$$

$$r = \frac{R}{H}z$$

$$dV = r^2\pi\, dz \quad /\cdot\int$$

$$V = \int_0^H \frac{R^2}{H^2} z^2 \pi\, dz = \frac{R^2\pi}{H^2}\frac{1}{3}z^3\Big|_0^H = \frac{R^2\pi}{\cancel{H^2}}\frac{1}{3}H^{\cancel{3}}$$

$$\boxed{V = \frac{1}{3}R^2\pi H}$$

$$z_c = \frac{1}{M}\int_{(M)} z\,dM = \frac{1}{M}\int_{(V)} z\rho\,dV = \frac{1}{\cancel{M}}\frac{\cancel{M}}{V}\int_{(V)} z\,dV =$$

$$= \frac{1}{V}\int_0^H z\,\frac{R^2}{H^2}z^2\pi\,dz = \frac{1}{V}\frac{R^2}{H^2}\pi\int_0^H z^3\,dz =$$

$$= \frac{1}{\frac{1}{3}R^2\cancel{\pi}H}\,\frac{\cancel{R^2}}{H^2}\,\cancel{\pi}\,\frac{1}{4}z^4\Big|_0^H = \boxed{\frac{3}{4}H}$$

$$J_z = \frac{1}{2}\int_{(M)} r^2\,dM = \frac{1}{2}\int_{(V)} r^2\rho\,dV = \frac{M}{2V}\int_{(V)} r^2\,dV = ⊛$$

$$dV = \frac{R^2}{H^2}z^2\pi\,dz$$

$$⊛ = \frac{M}{2V}\int_0^H \frac{R^2}{H^2}z^2\,\frac{R^2}{H^2}z^2\pi\,dz = \frac{M}{2V}\frac{R^4}{H^4}\pi\int_0^H z^4\,dz$$

$$= \frac{M}{2\frac{1}{3}\cancel{R^2}\cancel{\pi}\cancel{H}}\,\frac{R^{\cancel{4}\,2}}{\cancel{H^4}}\,\cancel{\pi}\,\frac{1}{5}\cancel{H^5} = \frac{3}{10}MR^2$$

$$J_{xy} = \int_{(M)} z^2\,dM = \int_{(V)} z^2\rho\,dV = \frac{M}{V}\int_0^H z^2\,\frac{R^2}{H^2}z^2\pi\,dz$$

$$= \frac{M}{\frac{1}{3}\cancel{R^2}\cancel{\pi}\cancel{H}}\,\frac{\cancel{R^2}}{\cancel{H^2}}\,\cancel{\pi}\,\frac{1}{5}H^{\cancel{5}\,2} = \frac{3MH^2}{5}$$

$$J_{xz} = J_{yz}$$

$$J_z = J_{xz} + J_{yz} = 2J_{xz} \Rightarrow J_{xz} = \frac{1}{2}J_z = \frac{3}{20}MR^2$$

$$J_x = J_{xz} + J_{xy} = \frac{3}{20} mR^2 + \frac{3}{5} mH^2 =$$

$$= \frac{3m}{20}\left(R^2 + 4H^2\right)$$

$$J_{x1} = J_x - m\left(\frac{3}{4}H\right)^2 = \frac{3m}{20}\left(R^2 + 4H^2\right) - \frac{9}{16} mH^2 =$$

$$= \frac{3m}{80}\left(4R^2 + 16H^2 - 15H^2\right) =$$

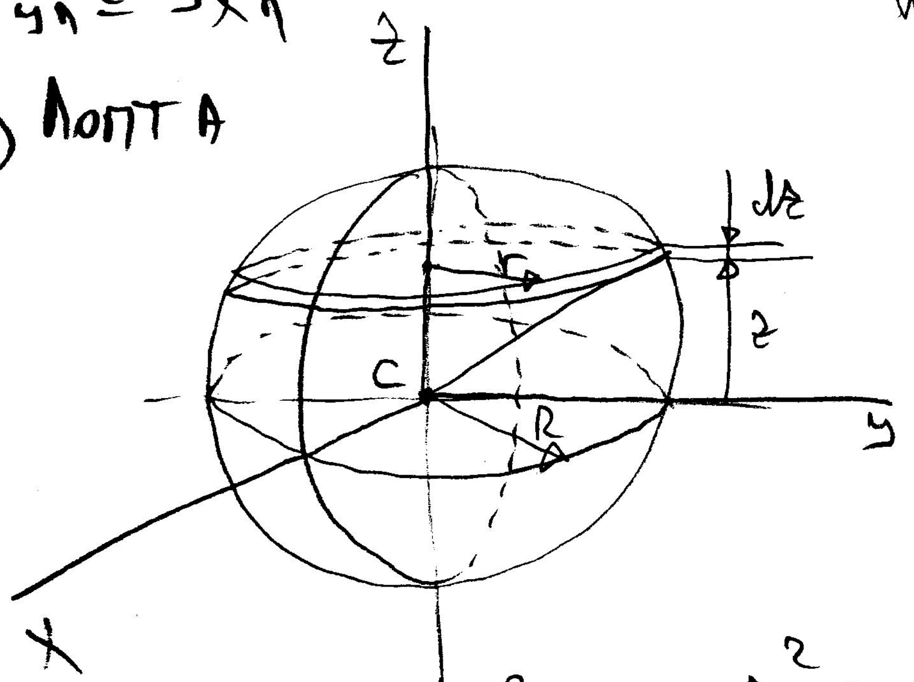
$$= \frac{3m}{80}\left(4R^2 + H^2\right)$$

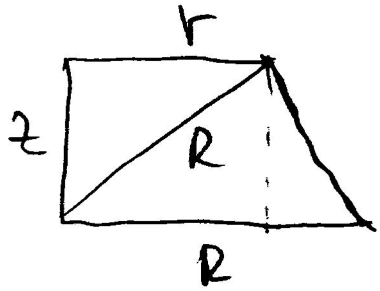
$$J_{y1} = J_{x1}$$

$m = \rho V, \quad \rho\left[\frac{kg}{m^3}\right]$

8) ЛОПТА

$dm = \rho\, dV$





$$dV = r^2\pi\, dz \quad \Big|\cdot\int$$

$$R^2 = z^2 + r^2$$
$$r^2 = R^2 - z^2$$

$$V = \int_{-R}^{R} r^2\pi\, dz =$$

$$= \pi\int_{-R}^{R}\left(R^2 - z^2\right) dz = \pi\left(R^2 z \Big|_{-R}^{R} - \frac{1}{3}z^3 \Big|_{-R}^{R}\right)$$

$$= \pi\left(2R^3 - \frac{2}{3}R^3\right) = \frac{4}{3}R^3\pi$$

$$J_{xy} = J_{xz} = J_{yz}$$

$$J_z = J_{xz} + J_{yz} = 2J_{xz} = 2J_{xy} = 2J_{yz}$$

$$J_{xy} = \int_{(m)} z^2\, dm = \int_{(V)} z^2 \rho\, dV = \frac{m}{V}\int_{-R}^{R} z^2 r^2 \pi\, dz =$$

$$= \frac{m\pi}{\frac{4}{3}R^3\pi}\int_{-R}^{R}(R^2 - z^2)z^2\, dz =$$

$$= \frac{3m}{4R^3}\left(R^2 \frac{1}{3} z^3 \Big|_{-R}^{R} - \frac{1}{5} z^5 \Big|_{-R}^{R} \right) =$$

$$= \frac{3m}{4R^3}\left(\frac{2}{3}R^5 - \frac{2}{5}R^5 \right) = \frac{m}{5}R^2$$

$$\boxed{J_z = \frac{2}{5} m R^2} = J_y = J_z$$